Composite Materials, PhD

Week 1

Macromechanical Analysis of a Lamina

Part 1: Deformations of a Unidirectional Lamina Under Applied Loads

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Introduction

In this lecture, we explored the macromechanical analysis of laminas, covering stress-strain relationships, stiffness and compliance matrices, and problem-based applications in composite engineering.

Objectives

By the end of this session, students will be able to:

- Understand the mechanical behavior of a unidirectional lamina.
- Analyze stress and strain relations in composite materials.
- Compute stiffness and compliance matrices.
- Apply stress transformation equations.
- Solve engineering problems related to composite laminas.

Topics to be covered

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- Deformation of Unidirectional Lamina
- Stress and Strain Analysis
- Elastic Moduli and Stiffness Matrices
- Compliance Matrices for Different Material Types
- Examples of Stress Analysis in Composite Laminas

Typical Laminate

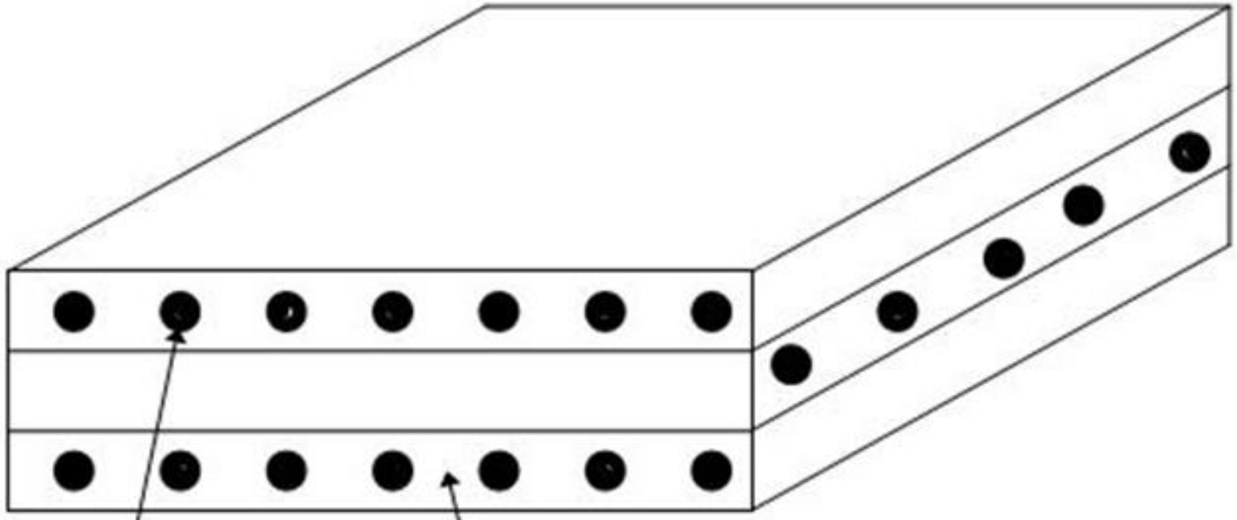

FIGURE 2.1
Typical laminate made of three laminas

Deformation of Unidirectional Lamina

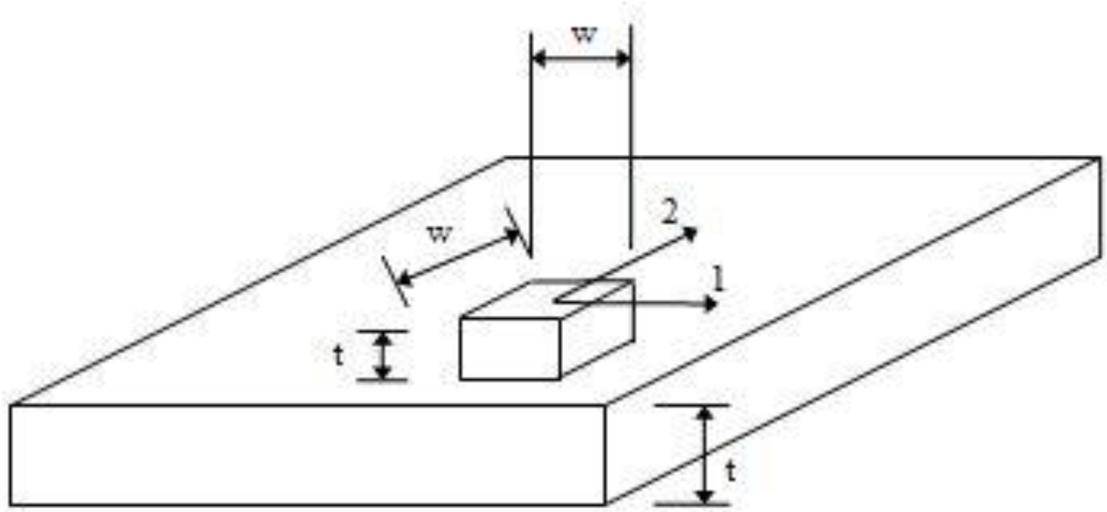

FIGURE 2.2
Deformation of square, isotropic plate under normal loads

Deformation of Unidirectional Lamina

$$\delta_{1A} = \delta_{2B},$$
$$\delta_{2A} = \delta_{1B}$$

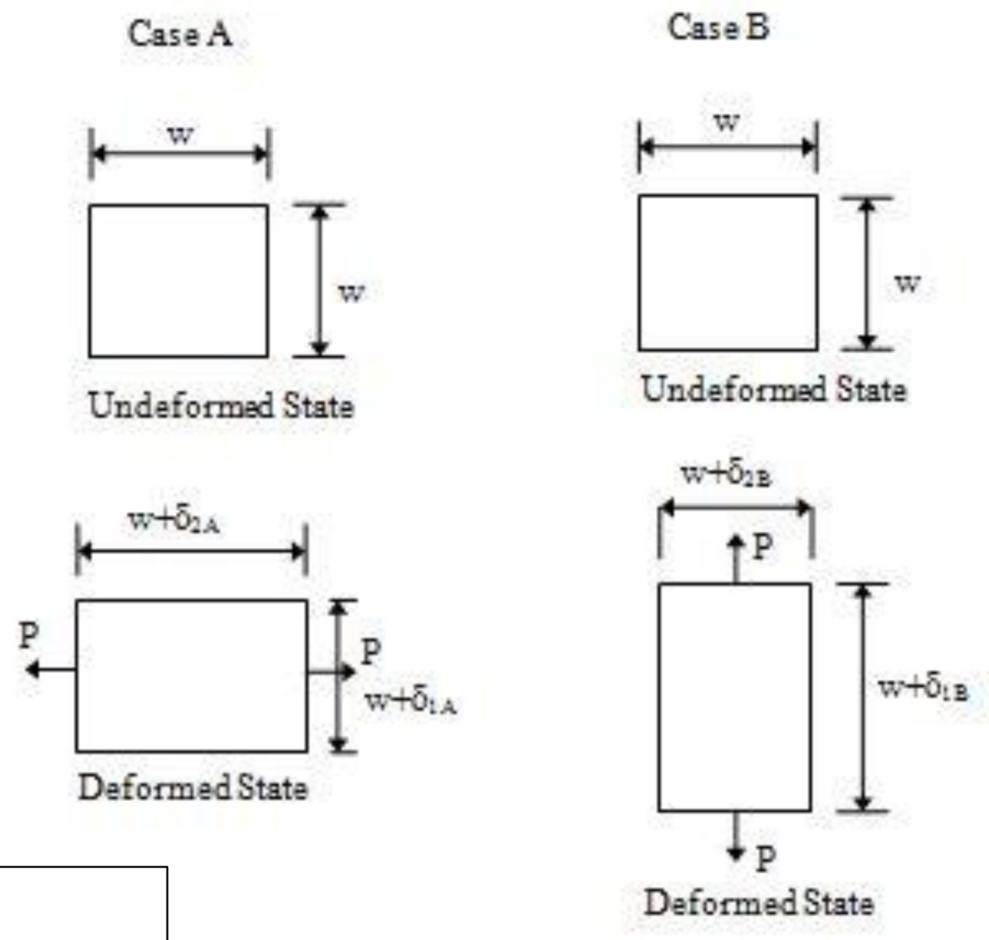

FIGURE 2.2
Deformation of square, isotropic plate under normal loads

Deformation of Unidirectional Lamina

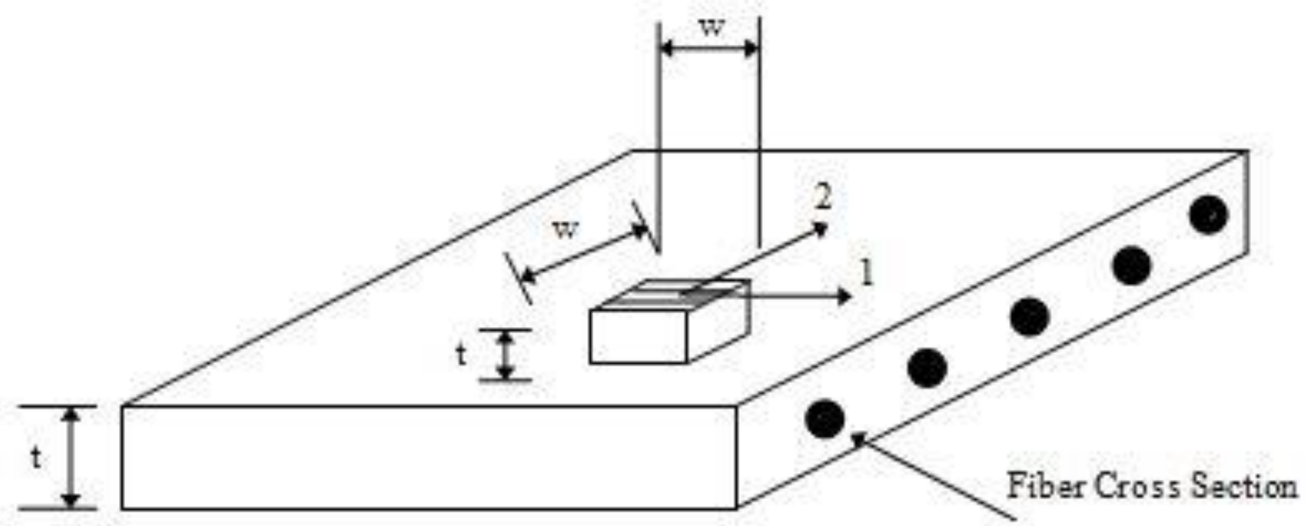

FIGURE 2.3
Deformation of square, unidirectional lamina with fibers at zero angle under normal loads

Deformation of Unidirectional Lamina

$$\delta_{1A} \neq \delta_{2B},$$

$$\delta_{2A} \neq \delta_{1B}$$

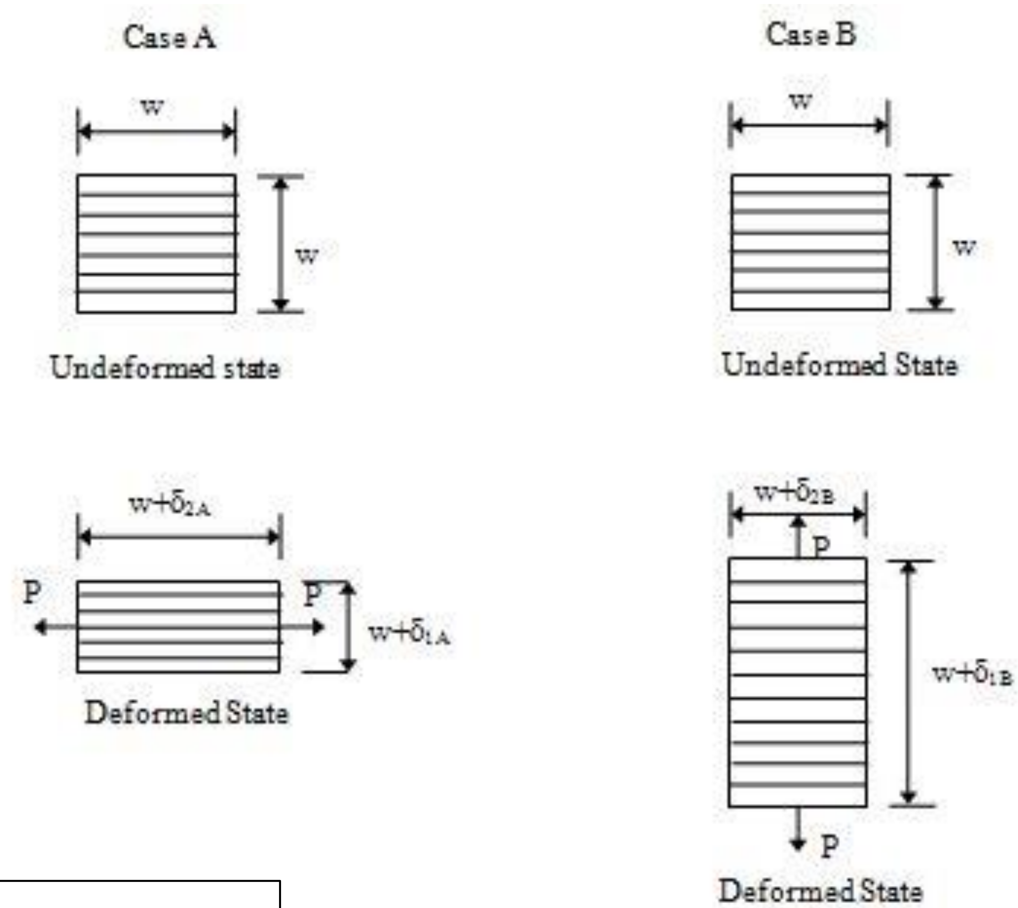

FIGURE 2.2
Deformation of square, unidirectional lamina
With fibers at zero angle under normal loads.

Deformation of Unidirectional Lamina

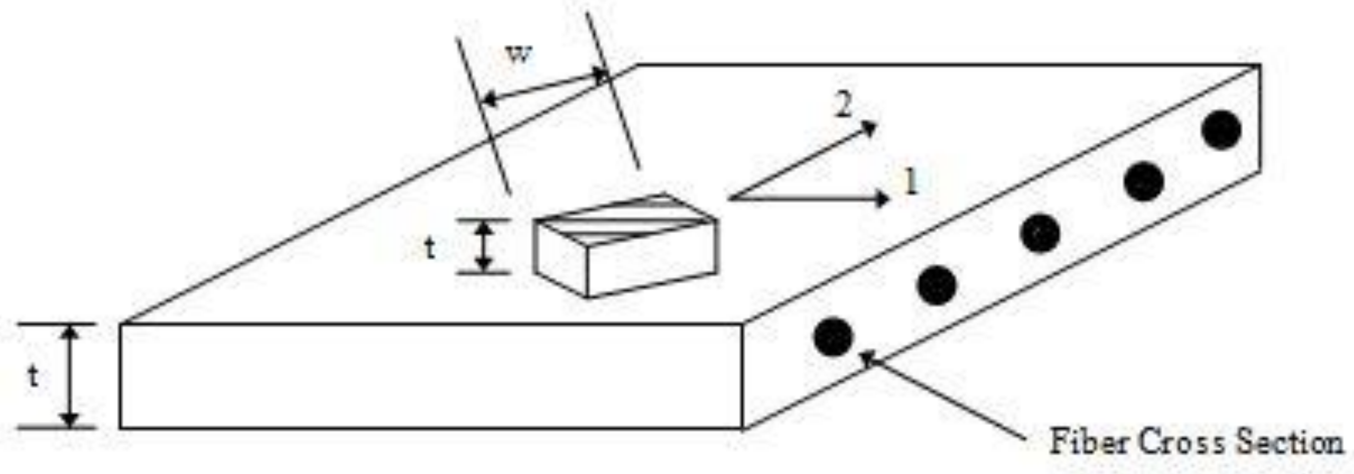

FIGURE 2.4
Deformation of square, unidirectional lamina with fibers at an angle to normal loads

Deformation of Unidirectional Lamina

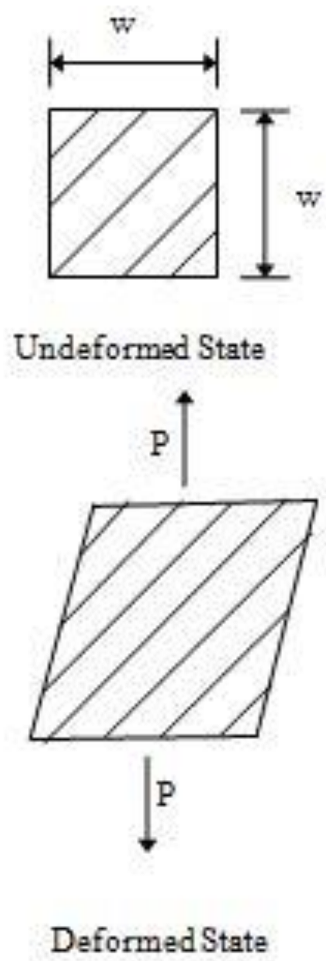

FIGURE 2.4
Deformation of square, unidirectional lamina with fibers at an angle to normal loads.

Stress

$$\sigma_n = \lim_{\Delta A \to 0} \frac{\Delta P_n}{\Delta A},$$

$$\tau_s = \lim_{\Delta A \to 0} \frac{\Delta P_s}{\Delta A}$$

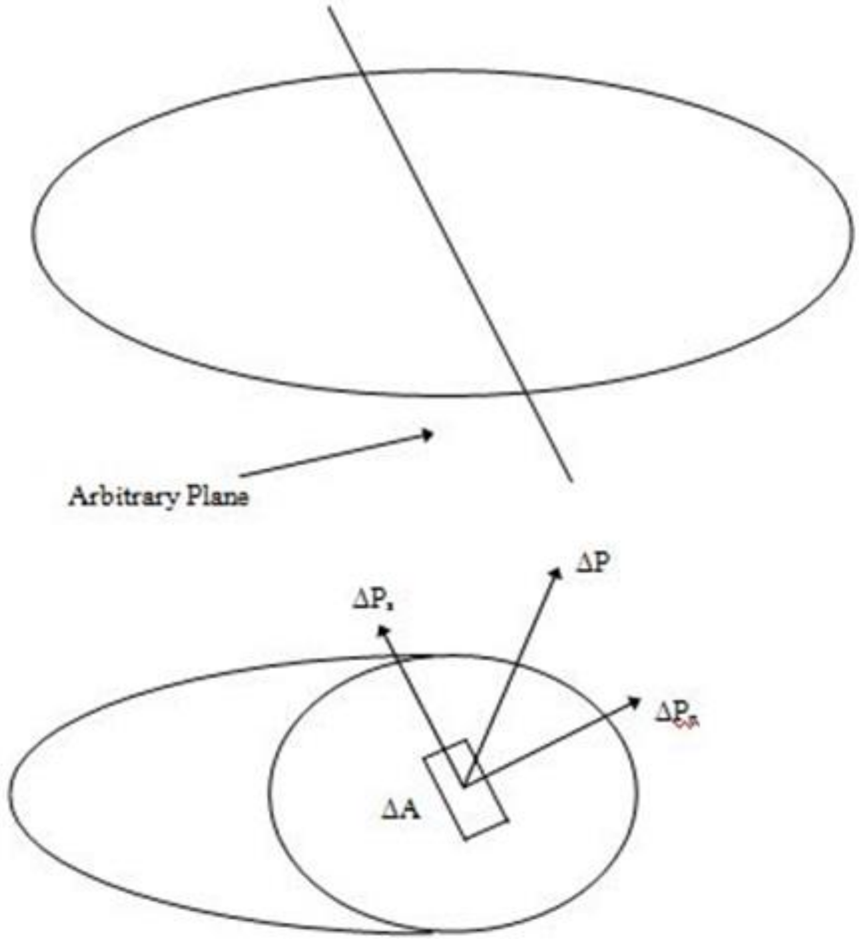

FIGURE 2.5
Stresses on infinitesimal area on an arbitrary plane

Stress

$$\sigma_x = \lim_{\Delta A \to 0} \frac{\Delta P_x}{\Delta A},$$

$$\tau_{xy} = \lim_{\Delta A \to 0} \frac{\Delta P_y}{\Delta A},$$

$$\tau_{xz} = \lim_{\Delta A \to 0} \frac{\Delta P_z}{\Delta A}$$

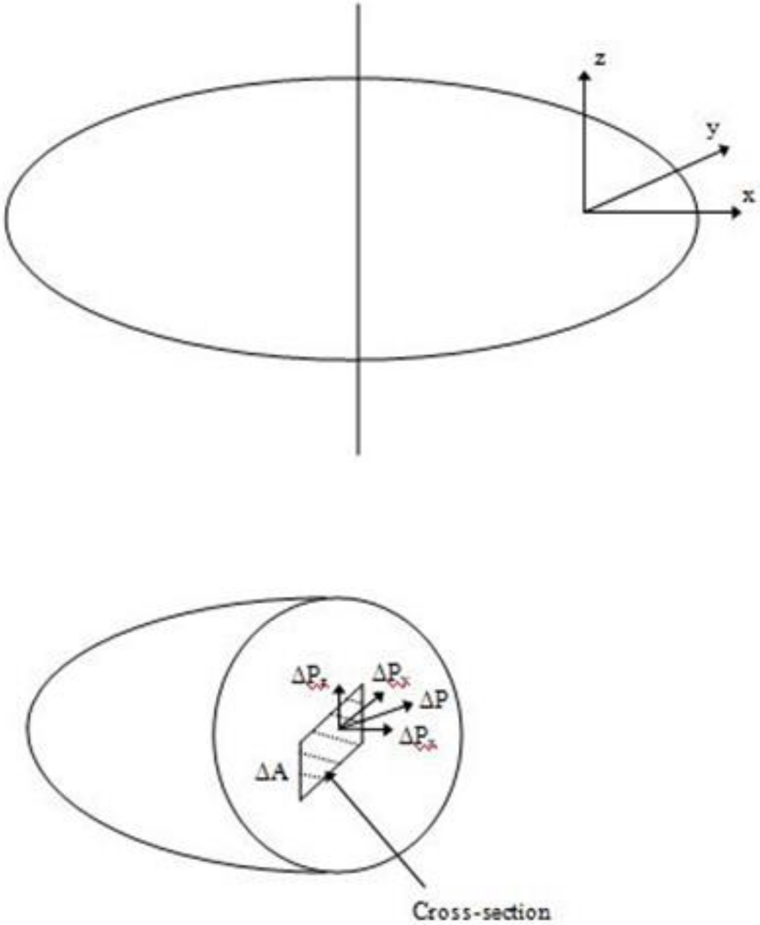

FIGURE 2.6
Forces on an infinitesimal area on the y-z plane

Stress

$$\tau_{xy} = \tau_{yx},$$

$$\tau_{yz} = \tau_{zy},$$

$$\tau_{zx} = \tau_{xz}$$

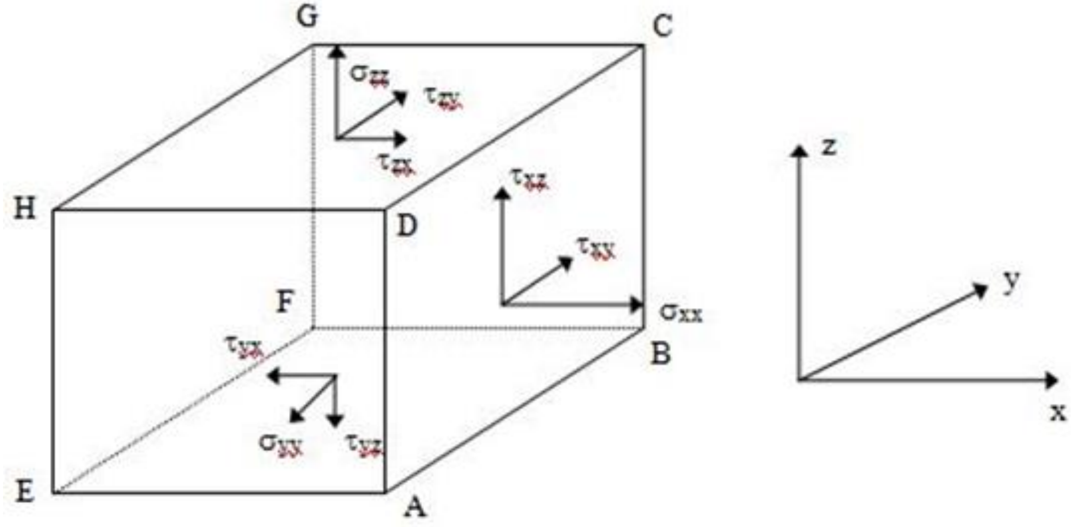

FIGURE 2.7
Stresses on an infinitesimal cuboid

Strain

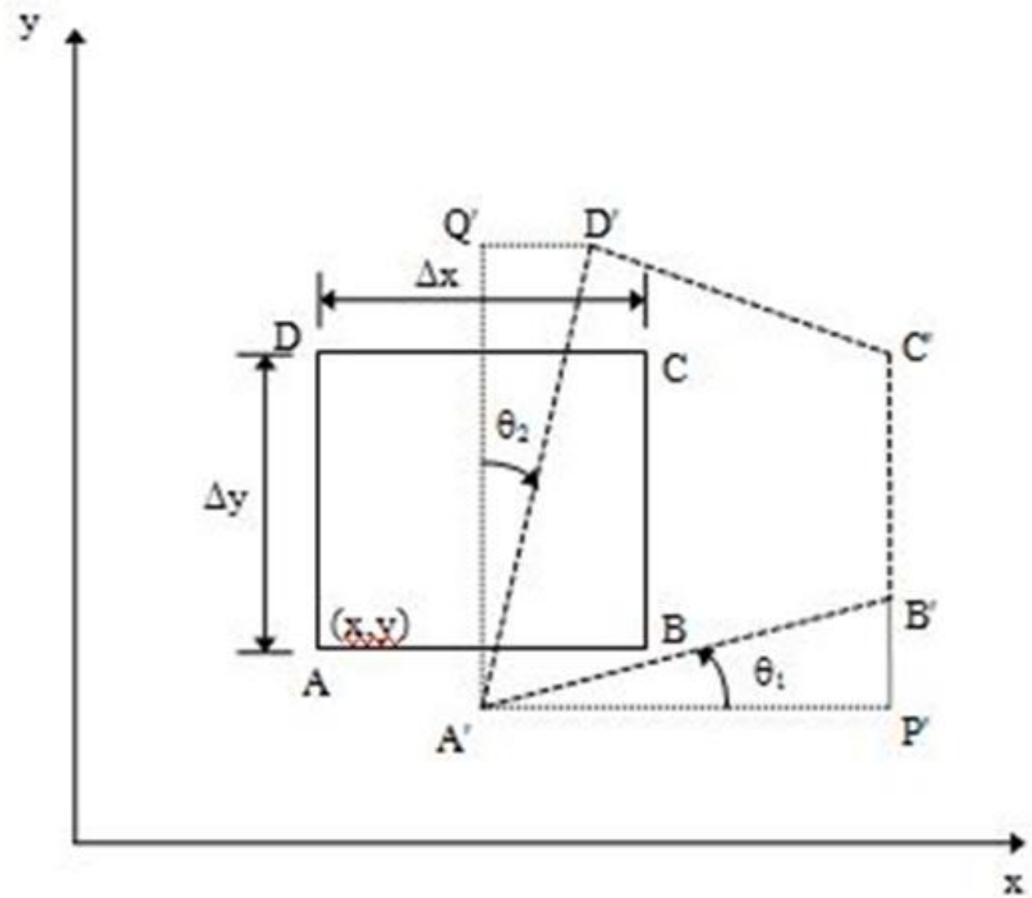

FIGURE 2.8
Normal and shearing strains on an infinitesimal area in the x-y plane

Strain

$u = u(x,y,z)$ = displacement in x-direction at point (x,y,z),

$v = v(x,y,z)$ = displacement in y-direction at point (x,y,z),

$w = w(x,y,z)$ = displacement in z-direction at point (x,y,z)

$$\varepsilon_x = \lim_{AB \to 0} \frac{A'B' - AB}{AB}$$

Where:

$$A'B' = \sqrt{(A'P')^2 + (B'P')^2}$$

$$= \sqrt{[\Delta x + u(x + \Delta x, y) - u(x,y)]^2 + [v(x + \Delta x, y) - v(x,y)]^2},$$

$$AB = \Delta x$$

Strain

Substituting

$$\varepsilon_x = \lim_{\Delta x \to 0} \left\{ \left[1 + \frac{u(x + \Delta +) - u(x,y)}{\Delta x} \right]^2 + \left[\frac{v(x + \Delta +) - v(x,y)}{\Delta x} \right]^2 \right\}^{1/2} -1$$

↓

$$\varepsilon_x = \left[\left(1 + \frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} \right)^2 \right]^{1/2} -1$$

$$\frac{\partial u}{\partial x} << 1 \longrightarrow$$

$$\frac{\partial v}{\partial x} << 1 \longrightarrow$$

$$\varepsilon_x = \frac{\partial u}{\partial x}$$

Strain

$u = u(x,y,z)$ = displacement in x-direction at point (x,y,z),

$v = v(x,y,z)$ = displacement in y-direction at point (x,y,z),

$w = w(x,y,z)$ = displacement in z-direction at point (x,y,z)

$$\varepsilon_y = \lim_{AD \to 0} \frac{A'D' - AD}{AD}$$

Where:

$$A'D' = \sqrt{(A'Q')^2 + (Q'D')^2}$$

$$= \sqrt{[\Delta y + v(x,y+\Delta,) - v(x,y)]^2 + [u(x,y+\Delta,) - u(x,y)]^2},$$

$$AD = \Delta y$$

Strain

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Substituting

$$\varepsilon_y = \lim_{\Delta y \to 0} \left\{ \left[1 + \frac{v(x, y+\Delta y) - v(x,y)\}}{\Delta y} \right]^2 + \left[\frac{u(x, y+\Delta y) - u(x,y)}{\Delta y} \right]^2 \right\}^{1/2} - 1$$

↓

$$\varepsilon_y = \left[\left(1 + \frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right]^{1/2} - 1$$

$$\frac{\partial u}{\partial y} << 1 \longrightarrow$$

$$\frac{\partial v}{\partial y} << 1 \longrightarrow$$

$$\varepsilon_y = \frac{\partial v}{\partial y}$$

Strain

$$\gamma_{xy} = \theta_1 + \theta_2$$

Where:

$$\theta_1 = \lim_{AB \to 0} \frac{P'B'}{A'P'},$$

$$P'B' = v(x + \Delta x, y) - v(x, y),$$

$$A'P' = u(x + \Delta x, y) + \Delta x - u(x, y)$$

Strain

$$\gamma_{xy} = \theta_1 + \theta_2$$

Where:

$$\theta_2 = \lim_{AD \to 0} \frac{Q'D'}{A'Q'},$$

$$Q'D' = u(x, y + \Delta y) - u(x, y),$$

$$A'Q' = v(x, y + \Delta y) + \Delta y - v(x, y)$$

Strain

Substituting

$$\gamma_{xy} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\dfrac{v(x+\Delta x, y) - v(x,y)}{\Delta x}}{\dfrac{u(x+\Delta x, y) + \Delta x - u(x,y)}{\Delta x}} + \frac{\dfrac{u(x, y+\Delta y) - u(x,y)}{\Delta y}}{\dfrac{v(x, y+\Delta y) + \Delta y - v(x,y)}{\Delta y}}$$

↓

$$\gamma_{xy} = \frac{\dfrac{\partial v}{\partial x}}{1+\dfrac{\partial u}{\partial x}} + \frac{\dfrac{\partial u}{\partial y}}{1+\dfrac{\partial u}{\partial y}}$$

$$\frac{\partial u}{\partial x} << 1 \longrightarrow$$

$$\frac{\partial v}{\partial y} << 1 \longrightarrow \qquad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

Strain

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y},$$

$$\gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z},$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z}$$

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \varepsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \qquad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

Elastic Moduli

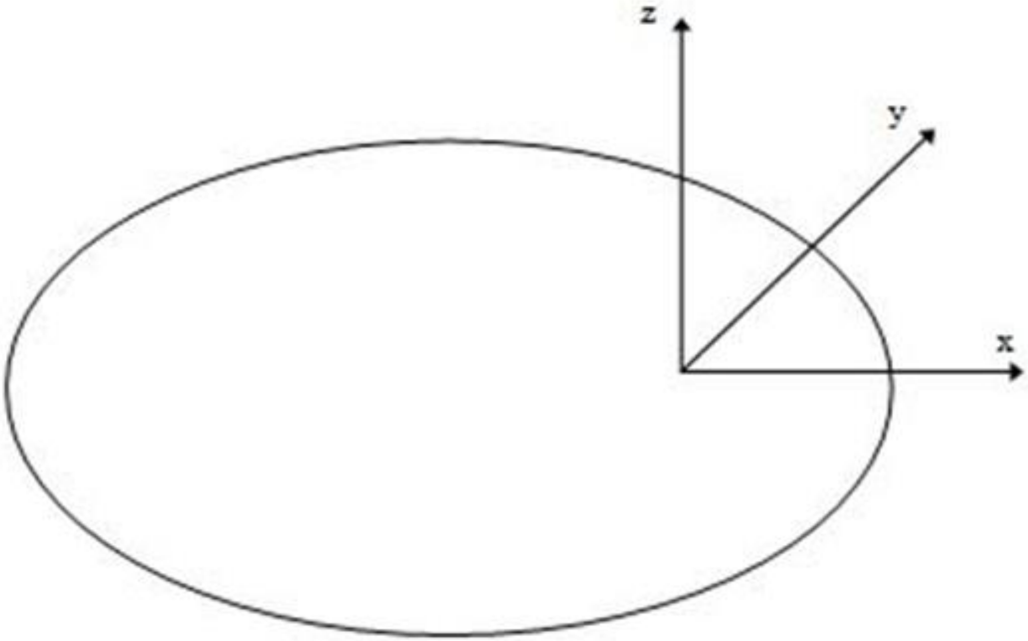

FIGURE 2.9
Cartesian coordinates in 3-D

Elastic Moduli

$$
\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{bmatrix} =
\begin{bmatrix}
\frac{1}{E} & -\frac{v}{E} & -\frac{v}{E} & 0 & 0 & 0 \\
-\frac{v}{E} & \frac{1}{E} & -\frac{v}{E} & 0 & 0 & 0 \\
-\frac{v}{E} & -\frac{v}{E} & \frac{1}{E} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{G}
\end{bmatrix}
\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{bmatrix}
$$

Elastic Moduli

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{yz}\\ \tau_{zx}\\ \tau_{xy}\end{bmatrix} = \begin{bmatrix} \frac{E(1-\nu)}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & 0 & 0 & 0 \\ \frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{E(1-\nu)}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & 0 & 0 & 0 \\ \frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{E(1-\nu)}{(1-2\nu)(1+\nu)} & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix} \begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{yz}\\ \gamma_{zx}\\ \gamma_{xy}\end{bmatrix}$$

Where:

$$G = \frac{E}{2(1+\nu)}$$

Strain Energy

$$W = \frac{1}{2}(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{xy} \gamma_{xy} + \tau_{yz} \gamma_{yz} + \tau_{zx} \gamma_{zx})$$

Example 2.1

A composite aerospace panel is subjected to an in-plane load due to aerodynamic forces. The panel consists of a unidirectional lamina with fibers aligned along the x-axis. Engineers need to determine the deformation behavior under applied stresses to ensure the safety of the structure.

- Given Data:
  - Normal stress in fiber direction: $\sigma_x = 50$ MPa
  - Shear stress: $\tau_{xy} = 10$ MPa
  - Elastic properties of the composite material:
    - $E_1 = 150$ GPa (Longitudinal modulus)
    - $E_2 = 10$ GPa (Transverse modulus)
    - $G_{12} = 5$ GPa (Shear modulus)
    - $\nu_{12} = 0.25$ (Poisson's ratio)

Using the compliance matrix equation for an orthotropic lamina:

$$\varepsilon_x = \frac{1}{E_1}\sigma_x - \frac{\nu_{12}}{E_1}\sigma_y$$

$$\gamma_{xy} = \frac{1}{G_{12}}\tau_{xy}$$

Since $\sigma_y = 0$:

$$\varepsilon_x = \frac{1}{150} \times 50 - \frac{0.25}{150} \times 0 = 0.000333$$

$$\gamma_{xy} = \frac{1}{5} \times 10 = 0.002 \text{ rad}$$

Thus, the normal strain in the fiber direction is 0.000333, and the shear strain is 0.002 rad.

Lamina and Laminate

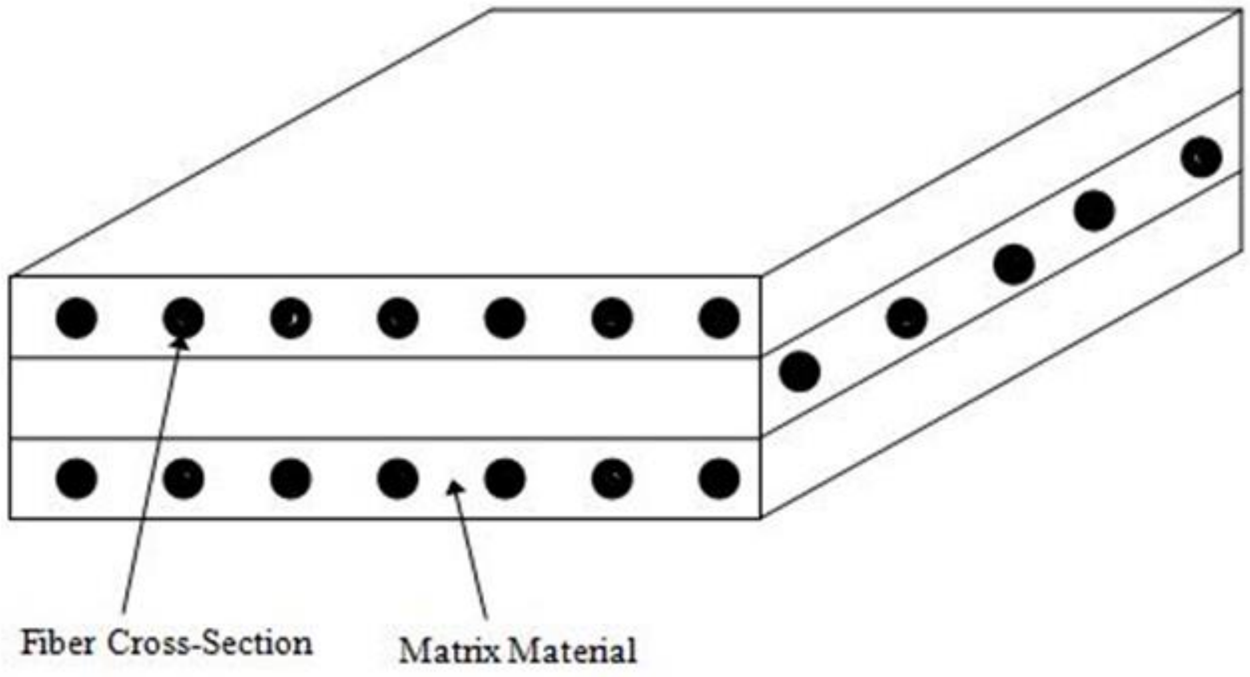

FIGURE 2.1
Typical laminate made of three laminas

Compliance Matrix [S] for General Material

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix}$$

Stiffness Matrix [C] for General Material

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix}$$

Stiffness matrix [C] has 36 constants

Compliance Matrix [S] for Isotropic Materials

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$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{11} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{12} & S_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(S_{11}-S_{12}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(S_{11}-S_{12}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(S_{11}-S_{12}) \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix}$$

Stiffness Matrix [C] for Isotropic Materials

$$
\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix}
=
\begin{bmatrix}
C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\
C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\
C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\
0 & 0 & 0 & 2(C_{11}-C_{12}) & 0 & 0 \\
0 & 0 & 0 & 0 & 2(C_{11}-C_{12}) & 0 \\
0 & 0 & 0 & 0 & 0 & 2(C_{11}-C_{12})
\end{bmatrix}
\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix}
$$

Compliance Matrix [S] for Isotropic Materials

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix}$$

Stiffness Matrix [C] for Isotropic Materials

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{yz}\\ \tau_{zx}\\ \tau_{xy}\end{bmatrix}=\begin{bmatrix}\frac{E(1-\nu)}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & 0 & 0 & 0\\ \frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{E(1-\nu)}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & 0 & 0 & 0\\ \frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{E(1-\nu)}{(1-2\nu)(1+\nu)} & 0 & 0 & 0\\ 0 & 0 & 0 & G & 0 & 0\\ 0 & 0 & 0 & 0 & G & 0\\ 0 & 0 & 0 & 0 & 0 & G\end{bmatrix}\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{yz}\\ \gamma_{zx}\\ \gamma_{xy}\end{bmatrix},$$

Compliance Matrix [S] for Anisotropic Material

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix}$$

Stiffness Matrix [C] for Anisotropic Material

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix}$$

Stiffness matrix [C] has 36 constants

Example 2.2

A structural engineer is designing a carbon-fiber-reinforced panel for an automobile chassis. To optimize the mechanical performance,

the stiffness matrix of the lamina must be determined..

- Given Data:
  - $E_1 = 150$ GPa
  - $E_2 = 10$ GPa
  - $G_{12} = 5$ GPa
  - $\nu_{12} = 0.25$

The stiffness matrix $[C]$ is given by:

$$[C] = \begin{bmatrix} \frac{E_1}{1-\nu_{12}\nu_{21}} & \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} & 0 \\ \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} & \frac{E_2}{1-\nu_{12}\nu_{21}} & 0 \\ 0 & 0 & G_{12} \end{bmatrix}$$

Since $\nu_{21} = \frac{\nu_{12}E_2}{E_1} = \frac{0.25\times10}{150} = 0.0167$, we substitute:

$$[C] = \begin{bmatrix} \frac{150}{1-(0.25\times0.0167)} & \frac{0.25\times10}{1-(0.25\times0.0167)} & 0 \\ \frac{0.25\times10}{1-(0.25\times0.0167)} & \frac{10}{1-(0.25\times0.0167)} & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$[C] = \begin{bmatrix} 150.63 & 2.51 & 0 \\ 2.51 & 10.04 & 0 \\ 0 & 0 & 5 \end{bmatrix} \text{ GPa}$$

This matrix is crucial for predicting the composite panel's performance under load.

Monoclinic Materials

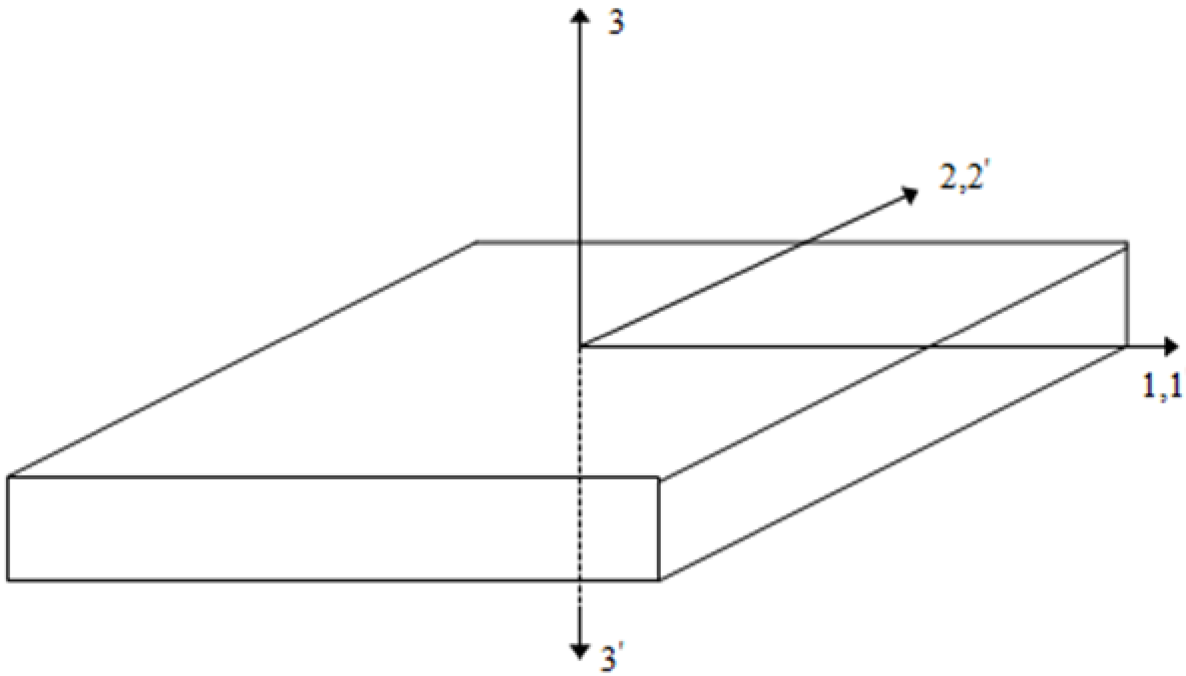
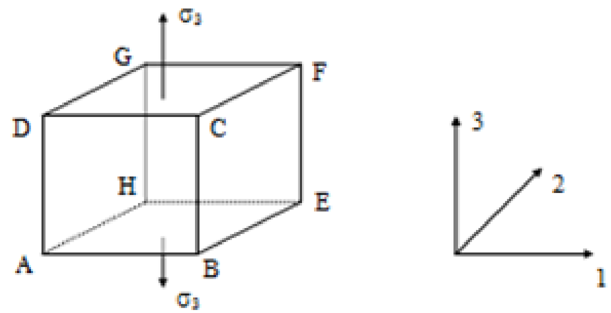


FIGURE 2.11
Transformation of coordinate axes for 1-2 plane of symmetry for a monoclinic material

Monoclinic Materials



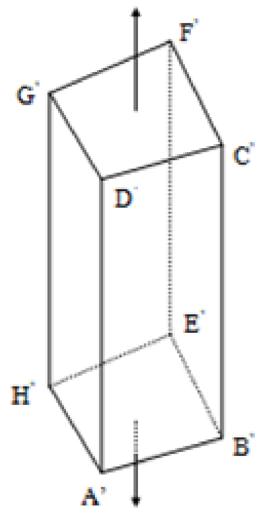


FIGURE 2.12
Deformation of a cubic element made of monoclinic material

Monoclinic Materials

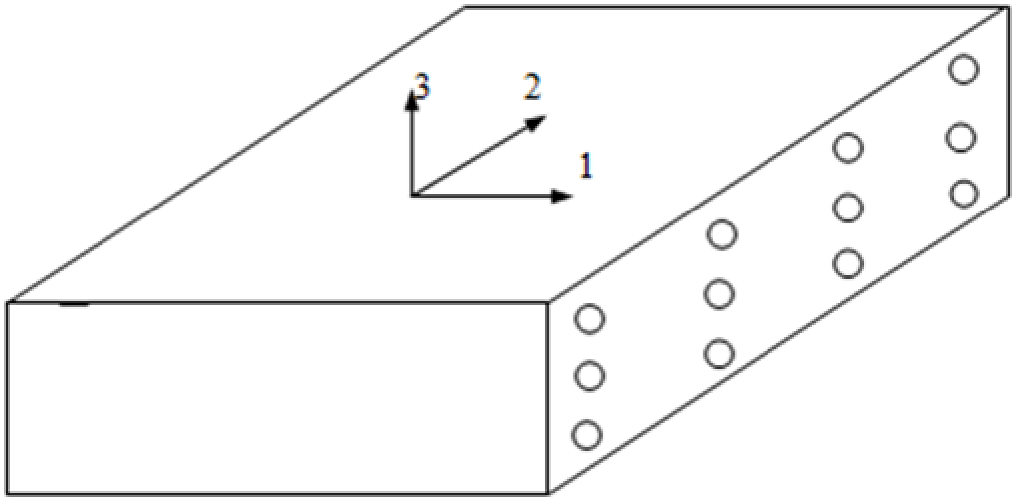


FIGURE 2.13
A unidirectional lamina as a monoclinic material with fibers arranged in a rectangular array

Compliance Matrix [S] for Monoclinic Materials

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & S_{16} \\ S_{12} & S_{22} & S_{23} & 0 & 0 & S_{26} \\ S_{13} & S_{23} & S_{33} & 0 & 0 & S_{36} \\ 0 & 0 & 0 & S_{44} & S_{45} & 0 \\ 0 & 0 & 0 & S_{45} & S_{55} & 0 \\ S_{16} & S_{26} & S_{36} & 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix}$$

Stiffness Matrix [C] for Monoclinic Materials

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$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix}$$

Orthotropic Materials

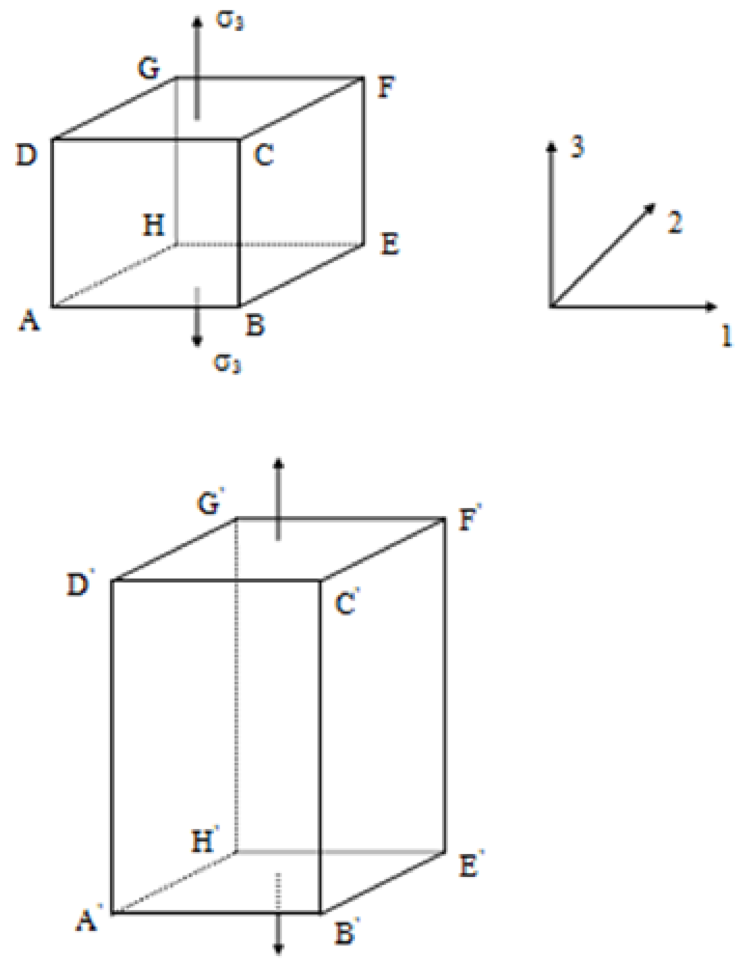


FIGURE 2.14
Deformation of a cubic element made of orthotropic material

Compliance Matrix [S] for Orthotropic Materials

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix}$$

Stiffness Matrix [C] for Orthotropic Materials

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix}$$

Compliance Matrix [S] for Orthotropic Materials

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & -\frac{\nu_{13}}{E_1} & 0 & 0 & 0 \\ -\frac{\nu_{21}}{E_2} & \frac{1}{E_2} & -\frac{\nu_{23}}{E_2} & 0 & 0 & 0 \\ -\frac{\nu_{31}}{E_3} & -\frac{\nu_{32}}{E_3} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{31}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix}$$

Stiffness Matrix [C] for Orthotropic Materials

$$
\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix}
=
\begin{bmatrix}
\frac{1-\nu_{23}\nu_{32}}{E_2E_3\Delta} & \frac{\nu_{21}+\nu_{23}\nu_{31}}{E_2E_3\Delta} & \frac{\nu_{31}+\nu_{21}\nu_{32}}{E_2E_3\Delta} & 0 & 0 & 0 \\
\frac{\nu_{21}+\nu_{23}\nu_{31}}{E_2E_3\Delta} & \frac{1-\nu_{13}\nu_{31}}{E_1E_3\Delta} & \frac{\nu_{32}+\nu_{12}\nu_{31}}{E_1E_3\Delta} & 0 & 0 & 0 \\
\frac{\nu_{31}+\nu_{21}\nu_{32}}{E_2E_3\Delta} & \frac{\nu_{32}+\nu_{12}\nu_{31}}{E_1E_3\Delta} & \frac{1-\nu_{12}\nu_{21}}{E_1E_2\Delta} & 0 & 0 & 0 \\
0 & 0 & 0 & G_{23} & 0 & 0 \\
0 & 0 & 0 & 0 & G_{31} & 0 \\
0 & 0 & 0 & 0 & 0 & G_{12}
\end{bmatrix}
\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix}
$$

Transversely Isotropic Materials

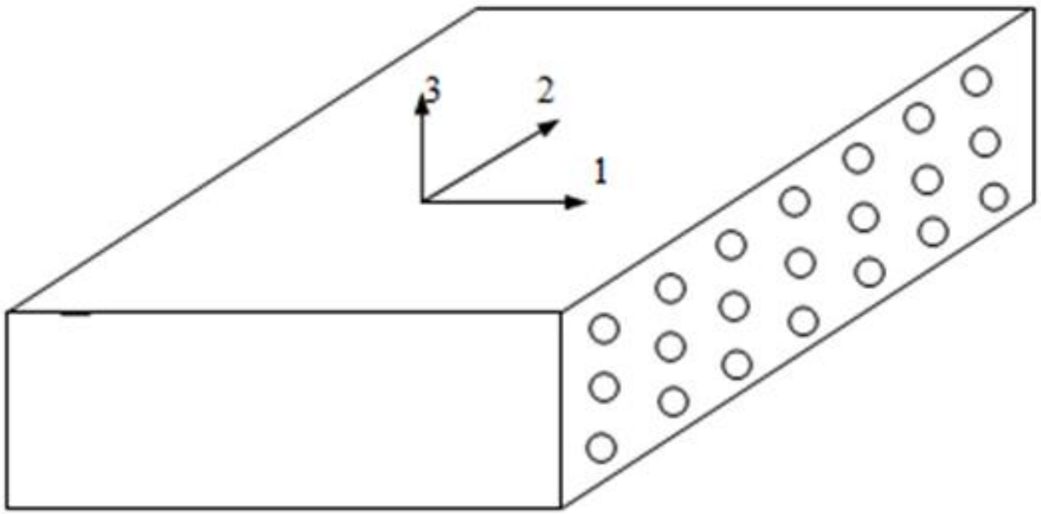


FIGURE 2.15
A unidirectional lamina as a transversely isotropic material with fibers arranged in a rectangular array

Compliance Matrix [S] for Transversely Isotropic Materials

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{12} & S_{23} & S_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(S_{22} - S_{23}) & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{55} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix}$$

Stiffness Matrix [C] for Transversely Isotropic Materials

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \tau_{23}\\ \tau_{31}\\ \tau_{12}\end{bmatrix}=\begin{bmatrix}C_{11} & C_{12} & C_{12} & 0 & 0 & 0\\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0\\ C_{12} & C_{23} & C_{22} & 0 & 0 & 0\\ 0 & 0 & 0 & \dfrac{C_{22}-C_{23}}{2} & 0 & 0\\ 0 & 0 & 0 & 0 & C_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & C_{55}\end{bmatrix}\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \gamma_{23}\\ \gamma_{31}\\ \gamma_{12}\end{bmatrix}$$

Compliance Matrix [S] for Isotropic Materials

$$
\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix} =
\begin{bmatrix}
S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\
S_{12} & S_{11} & S_{12} & 0 & 0 & 0 \\
S_{12} & S_{12} & S_{11} & 0 & 0 & 0 \\
0 & 0 & 0 & 2(S_{11}-S_{12}) & 0 & 0 \\
0 & 0 & 0 & 0 & 2(S_{11}-S_{12}) & 0 \\
0 & 0 & 0 & 0 & 0 & 2(S_{11}-S_{12})
\end{bmatrix}
\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix}
$$

Stiffness Matrix [C] for Isotropic Materials

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(C_{11}-C_{12}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(C_{11}-C_{12}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(C_{11}-C_{12}) \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix}$$

Compliance Matrix [S] for Isotropic Materials

$$
\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix} =
\begin{bmatrix}
\frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\
-\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\
-\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{G}
\end{bmatrix}
\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix}
$$

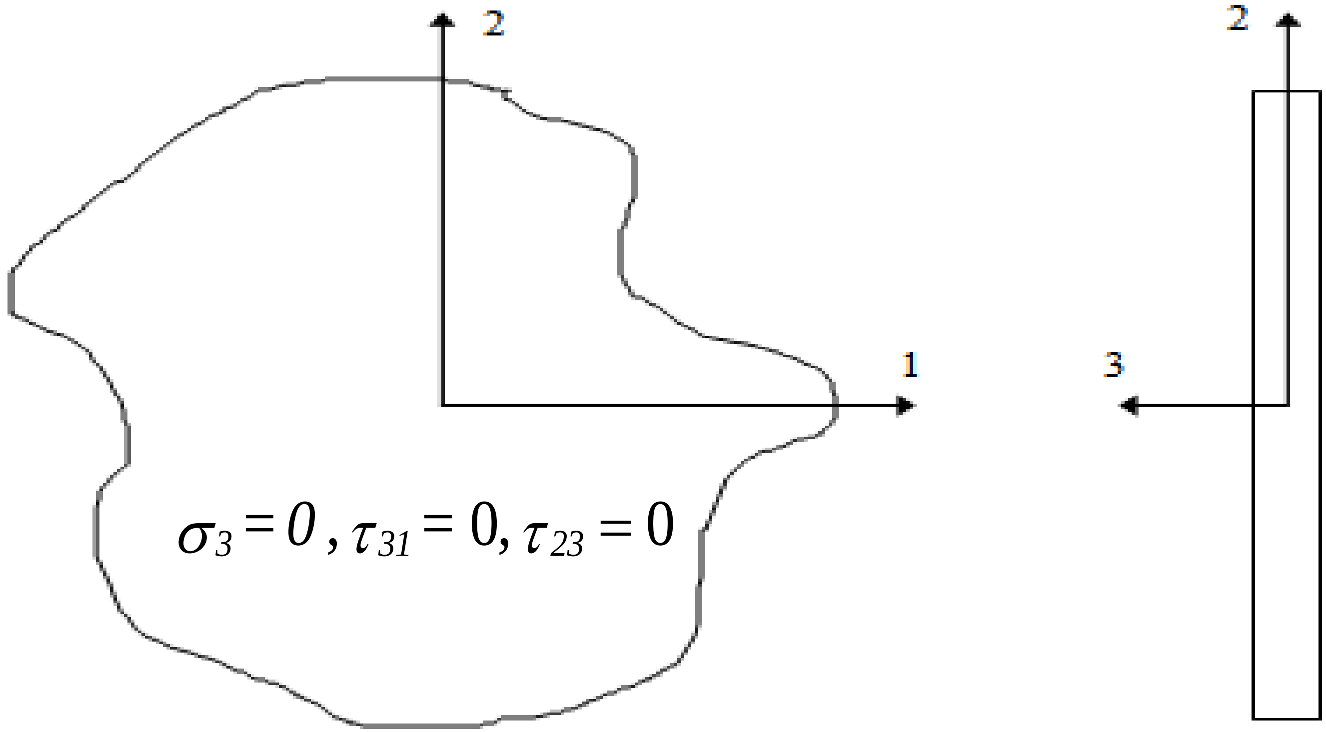
Stiffness Matrix [C] for Isotropic Materials

$$
\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{bmatrix} =
\begin{bmatrix}
\frac{E(1-\nu)}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & 0 & 0 & 0 \\
\frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{E(1-\nu)}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & 0 & 0 & 0 \\
\frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{E(1-\nu)}{(1-2\nu)(1+\nu)} & 0 & 0 & 0 \\
0 & 0 & 0 & G & 0 & 0 \\
0 & 0 & 0 & 0 & G & 0 \\
0 & 0 & 0 & 0 & 0 & G
\end{bmatrix}
\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{bmatrix},
$$

Independent Elastic Constants

Material Type	Independent Elastic Constants
Anisotropic	21
Monoclinic	13
Orthotropic	9
Transversely Isotropic	5
Isotropic	2

Plane Stress Assumption



- Upper and lower surfaces are free from external loads

$$\sigma_3 = 0, \tau_{23} = 0, \tau_{31} = 0,$$

FIGURE 2.17
Plane stress conditions for a thin plate

Example 2.3

A lightweight composite wing structure experiences stress due to aerodynamic forces. The principal material directions of the composite differ from the applied loading. Engineers need to transform the stresses to the principal material coordinates for accurate failure analysis.

- Given Data:
  - Orientation of the lamina: $\theta = 30^\circ$
  - Applied stresses:
    - $\sigma_x = 60$ MPa
    - $\tau_{xy} = 15$ MPa

Using the transformation equations:

$$\sigma_1 = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \sin\theta\cos\theta$$

$$\sigma_2 = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy} \sin\theta\cos\theta$$

$$\tau_{12} = (\sigma_x - \sigma_y)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

Substituting values:

$$\sigma_1 = 60 \times \cos^2 30^\circ + 0 \times \sin^2 30^\circ + 2 \times 15 \times \sin 30^\circ \cos 30^\circ$$

$$\sigma_1 = 60 \times 0.75 + 0 + 2 \times 15 \times 0.5 \times 0.866$$

$$\sigma_1 = 45 + 12.99 = 57.99 \text{ MPa}$$

$$\sigma_2 = 60 \times \sin^2 30^\circ + 0 \times \cos^2 30^\circ - 2 \times 15 \times \sin 30^\circ \cos 30^\circ$$

$$\sigma_2 = 60 \times 0.25 + 0 - 12.99$$

$$\sigma_2 = 15 - 12.99 = 2.01 \text{ MPa}$$

$$\tau_{12} = (60 - 0) \times \sin 30^\circ \cos 30^\circ + 15 \times (0.75 - 0.25)$$

$$\tau_{12} = 60 \times 0.5 \times 0.866 + 15 \times 0.5$$

$$\tau_{12} = 25.98 + 7.5 = 33.48 \text{ MPa}$$

Thus, the transformed stresses are:

- $\sigma_1 = 57.99$ MPa
- $\sigma_2 = 2.01$ MPa
- $\tau_{12} = 33.48$ MPa

This transformation is critical for evaluating composite materials under real loading conditions.

Summery

- The macromechanical analysis of a lamina

-Focusing on deformation under applied loads

-Stress-strain relations

-Stiffness/compliance matrices for composite materials.